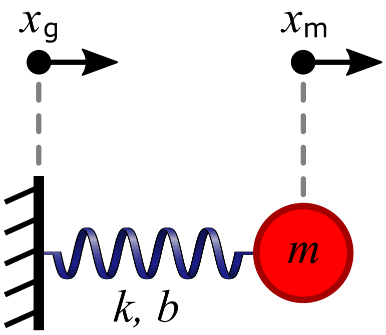
The inertial sensors relevant here are all based off mass-spring spring systems. Above resonance, the mass is 'stationary' and acts as an 'inertial reference' for the motion of its casing, referred to here as the 'ground'. On resonance, motion is amplified, and below resonance it is suppressed. However, by knowing the response of the mass-spring system, it is possible to invert its response and recover the ground motion at all frequencies. The signal size diminishes as f^2 below resonance, and it is eventually overwhelmed by noise.

Equations of motion and definitions



For the mass-spring system above, there is a restoring force proportional to the compression of the spring and a damping term proportional to the rate of change of the compression of the spring (assuming standard 'velocity' damping). We define the compression of the spring as the difference between ground and mass:

$$\Delta x = x_{\mathrm{m}} - x_{\mathrm{g}}.$$

For a spring constant k (N/m), with damping b (N/m/s), the equation of motion is:

$$m\ddot{x}_{\mathrm{m}} = -b\dot{\Delta x} - k\Delta x, \quad (1)$$
$$m\ddot{x}_{\mathrm{m}} = -b(\dot{x}_{\mathrm{m}} - \dot{x}_{\mathrm{g}}) - k(x_{\mathrm{m}} - x_{\mathrm{g}}). \quad (2)$$

The spring constant and mass determine the resonant frequency:

$$\omega_0 = \sqrt{k/m}\,(= 2\pi f_0).$$

The damping coefficient is combined with the mass and resonant frequency to make the (dimensionless) damping ratio, ζ, and quality factor, Q. They can also be used to produce the damping rate, defined as α in the 'pole-zero' note, which has units of angular frequency:

$$\zeta = \frac{1}{2Q} = \frac{b}{2m\omega_0},$$
$$\alpha = \frac{\omega_0}{2Q} = \omega_0\zeta.$$

Note that it is useful to rearrange the first of these to $\frac{b}{m} = \frac{\omega_0}{Q}$.The poles of the mass-spring response (see the 'pole-zero' technical note) are given by:

$$p = \alpha \pm \sqrt{\alpha^2 - \omega_0^2} = \omega_0(\zeta \pm \sqrt{\zeta - 1}).$$

In this form it is easy to see that if ζ is less than 1, the system is under-damped, and it is defined by a conjugate pair of roots. Note that this is equal to a Q of 0.5.

Transfer functions

To solve Eqn. **??** we take the Fourier transform so that the derivative property converts all the time-derivatives terms into algebraic terms of $i\omega$. Here we denote transformed variables with upper case letters, such that the Fourier transform of $x(t)$ is $X(\omega)$, and the equation of motion becomes:

$$-m\omega^2 X_{\mathrm{m}} = -i\omega b(X_{\mathrm{m}} - X_{\mathrm{g}}) - k(X_{\mathrm{m}} - X_{\mathrm{g}}).$$

Re-arranging we find the ratio:

$$\frac{X_{\rm m}(\omega)}{X_{\rm g}(\omega)} = \frac{i\omega b + k}{k - m\omega^2 + i\omega b}, = \frac{\omega_0^2 + \frac{i\omega\omega_0}{Q}}{\omega_0^2 - \omega^2 + \frac{i\omega\omega_0}{Q}}. \quad (3)$$

Since our inertial sensor can only measure the difference between the mass and the ground, it is useful to solve Eqn **??** directly in these terms. Subtracting $m\ddot{x}_{\rm g}$ from both sides and taking the Fourier transform we find:

$$-\omega^2 m X_{\rm g} = m\omega^2 \Delta X - i\omega b \Delta X - k\Delta X$$

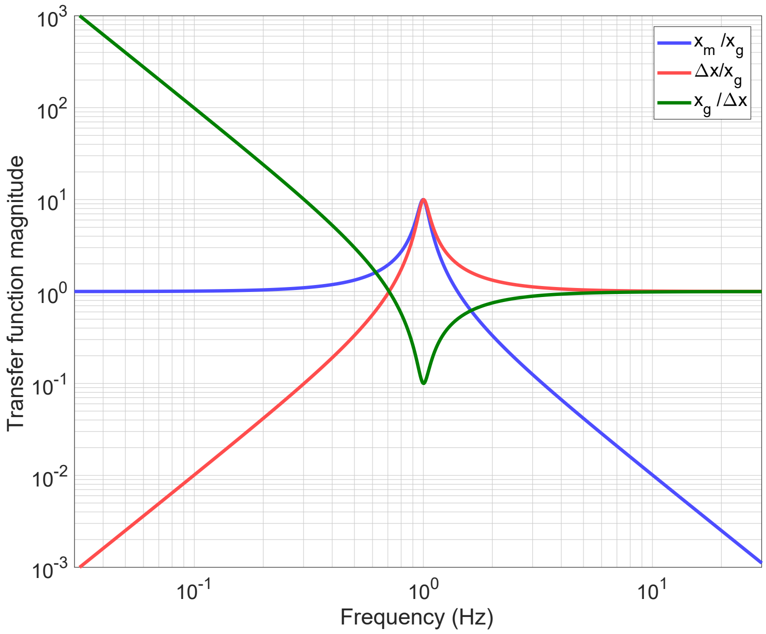
With some algebra, we find the response of the sensor to input ground motion:

$$\frac{\Delta X}{X_{\rm g}} = \frac{-\omega^2}{\omega_0^2 - \omega^2 + \frac{i\omega\omega_0}{Q}}.$$

To extract the ground motion, we simply need to invert this function and multiply it by the sensor output. For completeness, the inertial sensor plant-inversion is:

$$\frac{X_{\rm g}}{\Delta X} = -\frac{\omega_0^2 - \omega^2 + \frac{i\omega\omega_0}{Q}}{\omega^2}.$$

The three relevant transfer functions are shown below:
- mass motion per unit ground motion $\frac{X_{\rm m}}{X_{\rm g}}$,
- sensor output per unit ground motion $\frac{\Delta X}{X_{\rm g}}$,
- ground motion per unit sensor output $\frac{X_{\rm g}}{\Delta X}$.



Response to forces on the reference mass