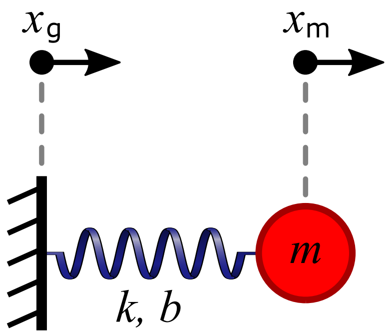
Previously we derived the response of a mass on a spring to input ground motion, and from there the response of the differential motion to ground motion. What we will do now is determine the effect of external forces acting on the proof mass and convert that into inertial-equivalent

Forced response of the mass



If we consider the ground to be fixed and apply an external force, f, to the 'mass' shown above, the equation of motion is:

$$m\ddot{x}_{\mathrm{m}} = f - b\dot{x_{\mathrm{m}}} - kx_{\mathrm{m}}. \tag{1}$$
$$\tag{2}$$

Taking the Fourier transform and rearranging we find:

$$-m\omega^2 X_{\mathrm{m}} = F - kX_{\mathrm{m}} - bX_{\mathrm{m}},$$

which, after substituting the resonant frequency and Q, leads to the transfer function from external force to displacement of the mass:

$$\frac{X_{\mathrm{m}}}{F} = \frac{1}{m}\frac{1}{\omega_0^2 - \omega^2 + \frac{i\omega\omega_0}{Q}}.$$

Transfer function to inertial output

A force acting on the mass displaces it according to Eqn. 3. This in turn changes the sensor output. The sensor output then passes through the plant inversion to convert it into ground motion, or in this case, equivalent ground motion. For a given external force, the apparent ground motion is:

$$X_{\mathrm{g,apparent}} = F\left[\frac{X_{\mathrm{m}}}{F}\right]\left[\frac{\Delta X}{X_{\mathrm{m}}}\right]\left[\frac{X_{\mathrm{g}}}{\Delta X}\right].$$

We have just derived the first of these, the second is equal to 1, and the third was calculated in the 'Sensor response' tutorial note. In total we find that the transfer function from external force to apparent ground motion is:

$$\frac{X_{\mathrm{g,apparent}}}{F} = \frac{-1}{m\omega^2},$$

which is exactly the same as the response of a free-mass to an external force. In other words, plant-inversion undoes the effect of the spring on the differential motion, allowing our 'bound' mass to impersonate a free mass.

Thermal driving forces

The thermal driving force, in Newtons per square-root Hertz, can be written as:

$$F_{\mathrm{th}} = \sqrt{4k_{\mathrm{B}}T\,\mathrm{Re}(Z(\omega))},$$

where k_{B} is Boltzmann's constant, T is the temperature (in Kelvin), and $\mathrm{Re}(Z(\omega))$ is the real part of the (frequency dependent) mechanical impedance, Z. The mechanical impedance is defined as the ratio of applied force, F, to resultant velocity of the mass, V_{m}, sort of like a spring-constant in velocity:

$$Z = \frac{F}{V_{\mathrm{m}}} = \frac{F}{i\omega X_{\mathrm{m}}}.$$

From our previous derivation, we already know $\frac{X_{\mathrm{m}}}{F}$. Inverting and multiplying by $i\omega$ gives:

$$Z = \frac{m\omega_0}{Q} - i\frac{m(\omega_0^2 - \omega^2)}{\omega}.$$

Note that our derivation assumed that the damping factor was frequency independent. For high-Q systems there should be no problem, but if the quality factor approaches 1, these solutions are not necessarily valid.

Structural damping

Through all of our derivation so far we have assumed some friction-like velocity-dependent damping. However, most high-Q materials exhibit frequency-independent 'structural' dissipation. To model this, the spring constant is replaced with a complex spring constant, $k = k_0(1 + i\phi)$, for $\phi << 1$. This means that the dissipation is now proportional to the displacement, but in-quadrature with the velocity. To convert, we can add some frequency dependence to our previous damping co-efficient, and we find that:

$$b_{\mathrm{str}} \quad = \quad \frac{k\phi}{\omega} = \frac{m\omega_0^2\phi}{\omega}.$$

For ready substitution into our previous calculation, we can use:

$$Q = \frac{\omega}{\omega_0\phi}.$$

All of this formalism can be found in Saulson, *Thermal noise in mechanical experiments*, PRD **42**, 2437 (1990).

Resulting inertial motion

The thermal force now becomes:

$$F_{\mathrm{th}} = \sqrt{4k_{\mathrm{B}}Tm\omega_0^2\frac{\phi}{\omega}}.$$

Converting this into apparent inertial motion using Eqn. 3, and ignoring the minus sign for this stochastic force, we find that the apparent motion induced by the thermal noise acting on the reference mass is:

$$X_{\mathrm{g,\,th}} = \sqrt{4k_{\mathrm{B}}Tm\omega_0^2\frac{\phi}{\omega}}\frac{1}{m\omega^2},$$

which is where the characteristic $\omega^{-5/2}$ slope comes from.